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Density Dependent Predator Death Prevalence Chaos in a Tri-Trophic Food Chain Model*

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Abstract. Ecological systems have all the properties to produce chaotic dynamics. To predict the chaotic behavior in an ecological system and its possible control mechanism is interesting. Aziz-Alaoui [1] considered a tri-trophic food-chain model with modified Leslie-Gower type growth rate for top-predator population and established the chaotic dynamics exhibited by the model system for a certain choice of parameter values. We have modified the said model by incorporating density dependent death rate for predator population. Our mathematical findings reveal the fact that there are two coexisting equilibrium points one of which is a source and the other one is a sink. The positive equilibrium point which is sink is actually globally asymptotically stable under certain parametric conditions. Numerical experiment analysis shows that the model system are capable to produce chaotic dynamics when the rate of intra specific completion is very low and chaotic dynamics disappears for a certain value of the rate of intra specific completion for predator species. Our results suggest that the consideration of density dependent death rate for predator species have the ability to control the chaotic dynamics.

Keywords: prey-predator, intra specific competition, stability, global stability, chaos, control.

1 Introduction

The dynamical problems associated with mathematical modeling of various ecological systems may appear to be simple at first sight but the detailed and systematic analysis often leads us to very interesting and fascinating dynamics. Specially with three or more dimensional ecological problems exposed within open environment have the potential to

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exhibit very complex dynamics compared to those observed in two dimensional models. In the natural world, dynamical behavior exhibited by various ecological systems are very complicated. For instance, the numbers of wild animals and plants are continuously variable quantities and these variations are quite irregular in period and also in amplitude [2]. Prince et al. [3] remarked that the study of community behaviour with the help of mathematical models must be based on at least three trophic levels and hence more focus should be made to study the complex behaviour exhibited by the deterministic models consist of three and more trophic levels. Present day scientists, not all of them, believe that "ecology is one of the basic discipline of chaos" [4]. Robert May [5] established the fact that a simple model for single species have the ability to produce chaotic dynamics, and theoretically demonstrated that the apparent stochastic oscillation of population could be explained by the highly complex and chaotic behavior arising from the simple deterministic mathematical models involving some nonlinear terms. A similar chaotic behavior is observed by Hastings and Powell [6] for a simple three dimensional food chain model within deterministic environment. After these pioneering works quite a good number of articles have appeared which demonstrate the chaotic dynamics observed in a wide range of three level food-chain models, e.g. see [2, 4, 7–15]. Chattopadhyay and Sarkar [16] observed that increasing the strength of toxic chemicals released by Toxin Producing Phytoplankton reduce the prevalence of chaos, in a tri-trophic food chain model. Recently Chatterjee et al. [17] have shown that chaos may also occur in eco-epidemiological systems. Various mathematical techniques, like local bifurcation analysis, global bifurcation analysis, extensive numerical simulations, Poincare-map etc. have been used to detect chaotic dynamics in multi-dimensional deterministic models of ecological system, but there is no unique mathematical tool to detect the parametric domain for which the model under consideration will exhibit chaotic oscillations and also what type of non-linear coupling terms are required to be present within the mathematical models which have potential to exhibit chaotic dynamics.

The control of chaotic dynamics with various ecological factors, namely, migration, predation, refuge, omnivory, habitat-heterogeneity is the most challenging task in such studies and thus it receives a good deal of attention from various scientists [18]. To the best of our knowledge, there are few literatures which have considered the inhibitory effect of "intra specific density dependence", but it has the ability to regulate the population dynamics significantly [19]. Based upon this idea we are intended to revisit the dynamical model for prey-specialist predator-generalist predator model system studied by Aziz-Alaoui [1] incorporating intra-specific density dependent death rate for specialist predator and show that the intra-specific density dependence has ability to control the chaotic oscillation observed in the original model.

The main objective of the paper is to extend the prey-predator-top predator model by incorporating density dependent death rate of specialist predator. We have analyzed the local asymptotic stability behaviour of the coexisting equilibrium points. Then, we have obtained the global stability conditions for the system. Finally, extensive numerical simulations is performed to make a further in depth analysis of the system. Finally, extensive numerical simulations are carried out to understand clearly the dynamics of the model system over a long time interval.

2 Basic mathematical model

In this paper we consider a three dimensional prey-predator model which consists of prey, specialist predator and generalist predator within deterministic environment. We assume $X \equiv X(t)$ is the population density of prey, $Y \equiv Y(t)$ is the population density of specialist predator and $Z \equiv Z(t)$ is that of generalist predator at any instant of time t. Here we consider the logistic law of growth for prey population, the interaction between prey and predator follows the Volterra scheme, i.e., specialist predator population die out exponentially in the presence of prey population and consumption of prey by specialist predator follows Holling type-II functional response. The interaction between specialist predator and generalist predator will follow the Leslie-Gower scheme [20–22], where the loss in generalist predator population is proportional to the reciprocal of per capita availability of its most favourite food and the growth of generalist predator depends upon the rate of mating between their male and female individuals. To study the effect of intra-specific density dependent death of specialist predator on the dynamical behaviour of prey-specialist predator-generalist predator model we consider the following system of nonlinear ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= a_0X - b_0X^2 - \frac{v_0XY}{d_0+X},\\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= \frac{v_1XY}{d_0+X} - a_1Y - dY^2 - \frac{v_2YZ}{d_1+Y},\\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= c_3Z^2 - \frac{v_3Z^2}{d_1+Y}
\end{aligned} \tag{1}$$

with initial conditions $X(0) \equiv X_0 \geq 0,\ Y(0) \equiv Y_0 \geq 0$ and $Z(0) \equiv Z_0 \geq 0$. Before going to discuss the ecological significance of the parameters involved with the model system we like to remark that the system resulting from system (1) with $d = 0$ is studied extensively by [1]. The parameters involved with the model (1) are positive and have ecological significance as follows: a_0 is the intrinsic birth rate of prey and b_0 stands for the intra-specific competition rate for them; v_0 is the maximum value of the per capita reduction rate of prey due to predation; d_0 measures the extent to which environment provides protection to prey population; v_1 is the growth rate of predator population; a_1 is the intrinsic death rate of predator population; d stands for the intra-specific competition rate of predator; v_2 is the removal rate of predator by super predators and d_1 is the half-saturation constant; c_3 is the growth rate of generalist predator due to mating between male and female individual (the number males and female individuals are assumed to be same); d_2 represents the residual loss of super predator population due to severe scarcity of its favorite food (in other words d_2 measures the extent to which environment provides protection to generalist predator population in absence of their favourite food source). Prey population of size X is the only source of food for specialist predator and specialist predator serve as a favourite food source for generalist predator. At this position we like to remark that first two equations of system (1) are quite standard and appeared in various well-known literature. In contrast the third equation is not a standard one and hence

before proceeding further we are intended to give some ecological justification behind the growth equation for generalist predator.

2.1 Ecological interpretation behind the third equation of system (1)

If we write the growth equation of generalist predator by following general Leslie-Gower scheme then it takes the following form

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \alpha Z\left(1 - \frac{Z}{\beta Y}\right). \tag{2}$$

It is based upon the idea that reduction in a generalist predator population has a reciprocal relationship with per capita availability of its preferred food, α is its intrinsic growth rate and its carrying capacity set by the environmental resources and is proportional to the abundance of its favourite food where β is the conversion factor of specialist predator into generalist predators. Generalist predator has opportunity to switch over from its favourite food to other available food sources when its favourite food source is not available in abundance. The consideration can be into account by adding a positive constant to the denominator of (2) as follows

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \alpha Z\left(1 - \frac{Z}{\beta Y + \gamma}\right) \tag{3}$$

and γ can be interpreted as the measure of carrying capacity for generalist predator population in absence of Y. Above equation can be rewritten as follows

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \alpha Z - \left(\frac{\alpha}{\beta}\right)\frac{Z^2}{Y + \frac{\gamma}{\beta}} \equiv c_3 Z - \frac{v_3 Z^2}{d_1 + Y}. \tag{4}$$

Finally, we get the third equation of system (1)

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = c_3 Z^2 - \frac{v_3 Z^2}{d_1 + Y}, \tag{5}$$

where $c_3 Z^2$ indicates the fact that mating frequency is directly proportional to the number of males and number females present at any instant of time t, and $v_3 = \alpha/\beta$, $d_1 = \gamma/\beta$ [14, 23].

The model can be simplified if it is written in dimensionless variables

$$X = \frac{a_0}{b_0}x, \quad Y = \frac{a_0^2}{b_0 v_0}y, \quad Z = \frac{a_0^3}{b_0 v_0 v_2}z, \quad T = \frac{t}{a_0},$$
$$\alpha = \frac{b_0 d_0}{a_0}, \quad \beta_1 = \frac{a_1}{a_0}, \quad \gamma_1 = \frac{v_1}{a_0},$$
$$\alpha_1 = \frac{d_1 v_0 b_0}{a_0^2}, \quad \beta_2 = \frac{c_3 a_0^2}{b_0 v_0 v_2}, \quad \gamma_2 = \frac{v_3}{v_2}, \quad \delta = \frac{d a_0}{b_0 v_0}.$$

Then, system (1) becomes

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= x - x^2 - \frac{xy}{\alpha + x}, \\
\frac{\mathrm{d}y}{\mathrm{d}\tau} &= \frac{\gamma_1 xy}{\alpha + x} - \beta_1 y - \delta y^2 - \frac{yz}{\alpha_1 + y}, \\
\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \beta_2 z^2 - \frac{\gamma_2 z^2}{\alpha_1 + y}
\end{aligned} \tag{6}$$

with $x(0) \equiv x_0 \geq 0$, $y(0) \equiv y_0 \geq 0$ and $z(0) \equiv z_0 \geq 0$.

2.2 Boundedness of the solutions

Lemma 1. *The positive cone* $\mathrm{Int}(R^3_+)$ *is invariant for system* (6)*.*

The proof is very simple and hence omitted.

To proof the boundedness of the solution of the system (6), we shall first recall the following comparison lemma given in [1].

Lemma 2. *Let* ϕ *be an absolutely continuous function satisfying the differential inequality:*

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} + k_1\phi(t) \leq k_2, \quad t \geq 0,$$

where $(k_1, k_2) \in \mathbf{R}^2, k_1 \neq 0$*. Then, for all* $t \geq \widetilde{T} \geq 0$*,*

$$\phi(t) \leq \frac{k_2}{k_1} - \left(\frac{k_2}{k_1} - \phi\big(\widetilde{T}\big)\right)e^{-k_1(t-\widetilde{T})}.$$

Theorem 1. *Let us assume*

$$\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1 < \frac{\gamma_2}{\beta_2}, \tag{7}$$

and let A *be the set defined by:*

$$\begin{aligned}
A = \Bigg\{(x, y, z) \in R^3_+ :\ & 0 \leq x \leq 1,\ \ 0 \leq x + \frac{y}{\gamma_1} \leq 1 + \frac{1}{4\beta_1}, \\
& 0 \leq x + \frac{y}{\gamma_1} + \alpha' z \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}\Bigg\},
\end{aligned}$$

where

$$\alpha' = \frac{1}{\beta_1^2\big(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1\big)} \quad \textit{and} \quad M = \frac{1}{4(\gamma_2 - \big(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1\big)\beta_2)}.$$

(i) *A is positively invariant;*

(ii) *all non-negative solutions (i.e. solutions initiating at R^3_+) of* (6) *are uniformly bounded forward in time, (thus they exists for all positive times), they eventually enter the attracting set A;*

(iii) *system* (6) *is dissipative.*

Proof. (i) Let $(x(0), y(0), z(0)) \in A$, obviously, from Lemma 1, $(x(t), y(t), z(t))$ remain non-negative; we will show that $(x(t), y(t), z(t)) \in A$ for all $t \geq 0$, we then have to prove that for all $t \geq 0$,

- Step (i-a): $x \leq 1$;
- Step (i-b): $x(t) + \dfrac{1}{\gamma_1} y(t) \leq 1 + \dfrac{1}{4\beta_1}$;
- Step (i-c): $x(t) + \dfrac{1}{\gamma_1} y(t) + \alpha' z(t) \leq 1 + \dfrac{1}{4\beta_1} + \dfrac{M}{\beta_1}$.

• Step (i-a). We first prove that $x(t) \leq 1$ for all $t \geq 0$. Since $x > 0,\ y > 0$ and $z > 0$ in $\mathrm{Int}(R^3_+)$, and solution $\phi(t) = (x(t), y(t), z(t))$ of (6), which starts in $\mathrm{Int}(R^3_+)$, satisfies the differential equation $\frac{dx}{dt} \leq x(1-x)$, this is obvious by considering the first equation of (6), moreover, due to the Lemma 1, these solution remain non-negative. Thus, $x(t)$ may be compared with solutions of

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = s(t)\big(1 - s(t)\big), \quad s(0) = x(0) > 0,$$

to get $x(t) \leq \frac{1}{1+ce^{-t}}$ for $t \geq 0$ (where $c = 1/x_0 - 1$). It follows that any non-negative solution $\phi(t)$ of (6) satisfies $x(t) \leq 1$ for all $t \geq 0$.

• Step (i-b). We now prove that $x(t) + \frac{1}{\gamma_1} y(t) \leq 1 + \frac{1}{4\beta_1}$ for all $t \geq 0$.
We define function $\sigma(t) = x(t) + \frac{1}{\gamma_1} y(t)$, the time derivative of which is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{\gamma_1}\frac{\mathrm{d}y}{\mathrm{d}t} = x(1-x) - \frac{\beta_1}{\gamma_1} y - \frac{\delta}{\gamma_1} y^2 - \frac{1}{\gamma_1}\frac{yz}{\alpha_1 + y}.$$

Since all parameters are positive, and solutions initiating in (R^3_+) remain in the non-negative cone then,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} \leq x(1-x) - \frac{\beta_1}{\gamma_1} y$$

holds for all x, y, and z non-negative. Thus,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} \leq x(1-x) + \beta_1 x - \beta_1\left(x + \frac{y}{\gamma_1}\right).$$

So

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} + \beta_1 \sigma(t) \leq \beta_1 + \frac{1}{4},$$

since in A, $0 \leq x \leq 1$ and $\max_{[0,1]}(x(1-x)) = \frac{1}{4}$. Using Lemma 2, we get, for all $t \geq \widetilde{T} \geq 0$,

$$\sigma(t) \leq 1 + \frac{1}{4\beta_1} - \left(1 + \frac{1}{4\beta_1} - \sigma(\widetilde{T})\right) e^{-\beta_1\left(t-\widetilde{T}\right)}, \tag{8}$$

then, if $\widetilde{T} = 0$,

$$\sigma(t) \leq 1 + \frac{1}{4\beta_1} - \left[1 + \frac{1}{4\beta_1} - \left(x(0) + \frac{y(0)}{\gamma_1}\right)\right] e^{-\beta_1(t-\widetilde{T})}.$$

Hence, since $(x(0), y(0), z(0)) \in A$, $x(t) + \frac{1}{\gamma_1} y(t) \leq 1 + \frac{1}{4\beta_1}$ for all $t \geq 0$.

• Step (i-c). We finally prove, provided $\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1 \leq \frac{\gamma_2}{\beta_2}$, that

$$0 \leq x + \frac{1}{\gamma_1} y + \alpha' z \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}$$

holds, if

$$\alpha' = \frac{1}{\beta_1^2\left(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1\right)} \quad \text{and} \quad M = \frac{1}{4\left(\gamma_2 - (\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1)\beta_2\right)}.$$

The proof is similar as done above; we define the following function:

$$\eta(t) = x(t) + \frac{1}{\gamma_1} y(t) + \alpha' z(t),$$

the time derivative of which is

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{\gamma_1}\frac{\mathrm{d}y}{\mathrm{d}t} + \alpha'\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= x(1-x) - \frac{\beta_1}{\gamma_1} y - \frac{\delta}{\gamma_1} y^2 - \frac{1}{\gamma_1}\frac{yz}{\alpha_1 + y} + \alpha'\left(\beta_2 - \frac{\gamma_2}{\alpha_1 + y}\right) z^2. \end{aligned}$$

Similarly to the previous Step (i-b), since every solution initiating in R^3_+ remains non-negative, all parameters are positive, $0 \leq x \leq 1$ and $max_{[0,1]}(x(1-x)) = \frac{1}{4}$, we get,

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} \leq \frac{1}{4} + \beta_1 - \beta_1\eta(t) + \alpha'\beta_1 z + \alpha'\left(\beta_2 - \frac{\gamma_2}{y + \alpha_1}\right) z^2,$$

thus, as in A, $y \leq \gamma_1 + \frac{\gamma_1}{4\beta_1}$, one gets

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} \leq \frac{1}{4} + \beta_1 - \beta_1\eta(t) + \alpha'\beta_1 z + \alpha'\left(\beta_2 - \frac{\gamma_2}{\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1}\right) z^2,$$

hence

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} + \beta_1\eta(t) \leq \frac{1}{4} + \beta_1 + M, \tag{9}$$

where

$$M = \max_{z \in R^+} \left(\alpha' \beta_1 z + \alpha' \Big(\beta_2 - \frac{\gamma_2}{\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1} \Big) z^2 \right).$$

The maximum M exists since, from the formula (7), one easily gets

$$\beta_2 - \frac{\gamma_2}{\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1} < 0,$$

and simple algebraic computations show that with

$$\alpha' = \frac{1}{\beta_1^2 \big(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1 \big)}, \quad M = \frac{1}{4\big(\gamma_2 - \big(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1 \big) \beta_2 \big)}.$$

Therefore, from equation (9), and using Lemma 2, we get for all $0 \geq x \geq 1$,

$$\eta(t) \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \left(1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \eta(\widetilde{T}) \right) e^{-\beta_1 (t - \widetilde{T})} \tag{10}$$

then, if $\widetilde{T} = 0$,

$$\eta(t) \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \left(1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \eta(0) \right) e^{-\beta_1 t}.$$

Thus since $(x(0), y(0), z(0)) \in A$,

$$x + \frac{1}{\gamma_1} y + \alpha' z \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}.$$

Consequently, we get the result

$$x + \frac{1}{\gamma_1} y + \frac{z}{\beta_1^2 \big(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1 \big)} \leq 1 + \frac{1}{4\beta_1} + \frac{1}{\beta_1} \frac{1}{4\big(\gamma_2 - \big(\gamma_1 + \frac{\gamma_1}{4\beta_1} + \alpha_1 \big) \beta_2 \big)}$$

for all $t \geq 0$.

(ii) We must prove that, for $(x(0), y(0), z(0)) \in R^3_+$, $(x(t), y(t), z(t)) \to A$ as $t \to \infty$.

We will follow the steps:

- Step (ii-a): $\limsup_{t \to \infty} x(t) \leq 1$;
- Step (ii-b): $\limsup_{t \to \infty} (x(t) + \frac{y(t)}{\gamma_1}) \leq 1 + \frac{1}{4\beta_1}$;
- Step (ii-c): $\limsup_{t \to \infty} (x(t) + \frac{y(t)}{\gamma_1} + \alpha' z(t)) \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}$.

• Step (ii-a). This results follows directly from Step (i-a) and Lemma 2, since solutions of the initial value problem $\frac{dx}{dt} = x(1-x),\ x(0) \geq 0$, satisfy $\lim\limits_{t\to\infty} \sup x(t) \leq 1$.

• Step (ii-b). Let $\epsilon \geq 0$ be given. Then there exists a $T_1 > 0$ such that $x(t) \leq 1 + \frac{\epsilon}{2}$ for all $t \geq T_1$. From (8) with $\widetilde{T} = T_1$, see Step (i-b), we get, for all $t \geq T_1 \geq 0$,

$$\begin{aligned}\sigma(t) = x(t) + \frac{y(t)}{\gamma_1} &\leq 1 + \frac{1}{4\beta_1} - \left[1 + \frac{1}{4\beta_1} - \left(x(T_1) + \frac{y(T_1)}{\gamma_1}\right)\right] e^{-\beta_1(t-T_1)} \\ &\leq 1 + \frac{1}{4\beta_1} - \left[\left(1 + \frac{1}{4\beta_1}\right)e^{\beta_1 T_1} - bigg(x(T_1) + \frac{y(T_1)}{\gamma_1}\right)e^{\beta_1 T_1}\Bigg] e^{-\beta_1 t} \\ &\leq 1 + \frac{1}{4\beta_1} - \left[\left(1 + \frac{1}{4\beta_1}\right) - \left(x(T_1) + \frac{y(T_1)}{\gamma_1}\right)e^{\beta_1 T_1}\right] e^{-\beta_1 t}.\end{aligned}$$

Then

$$x(t) + \frac{y(t)}{\gamma_1} \leq \left(1 + \frac{1}{4\beta_1} + \frac{\epsilon}{2}\right) - \left[\left(1 + \frac{1}{4\beta_1} + \frac{\epsilon}{2}\right) - \left(x(T_1) + \frac{y(T_1)}{\gamma_1}\right)e^{\beta_1 T_1}\right] e^{-\beta_1 t}.$$

for all $t \geq T_1$. Let $T_2 \geq T_1$ be such that

$$\left|\left(1 + \frac{1}{4\beta_1} + \frac{\epsilon}{2}\right) - \left(x(T_1) + \frac{y(T_1)}{\gamma_1}\right)e^{\beta_1 T_1}\right| e^{-\beta_1 t} \leq \frac{\epsilon}{2} \quad \text{for all} \quad t \geq T_2.$$

Then

$$x(t) + \frac{y(t)}{\gamma_1} \leq 1 + \frac{1}{4\beta_1} + \epsilon \quad \text{for all} \quad t \geq T_2.$$

Hence

$$\limsup_{t\to\infty} \left(x(t) + \frac{y(t)}{\gamma_1}\right) \leq 1 + \frac{1}{4\beta_1}.$$

• Step (ii-c). This proof is similar to the previous one. Let $\epsilon \geq 0$ be given. Then there exists a $T_3 \geq 0$ such that $x(t) + \frac{y(t)}{\gamma_1} \leq 1 + \frac{1}{4\beta_1} + \frac{\epsilon}{2}$ for all $t \geq T_3$.

From (10) with $\widetilde{T} = T_3$, see Step (i-c), we get, for all $t \geq T_3 \geq 0$,

$$\begin{aligned}\eta(t) = x(t) + \frac{1}{\gamma_1}y(t) + \alpha' z(t) &\\ \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \left[1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \eta(T_3)\right] e^{-\beta_1(t-T_3)} &\\ \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \left[\left(1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}\right)e^{\beta_1 T_3} - \eta(T_3)e^{\beta_1 T_3}\right] e^{-\beta_1 t} &\\ \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} - \left[\left(1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}\right) - \eta(T_3)e^{\beta_1 T_3}\right] e^{-\beta_1 t}. &\end{aligned}$$

Then

$$\eta(t) \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} + \frac{\epsilon}{2} - \left[1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} + \frac{\epsilon}{2} - \eta(T_3)e^{\beta_1 T_3}\right] e^{-\beta_1 t}.$$

Let $T_4 \geq T_3$ be such that

$$\left|1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} + \frac{\epsilon}{2} - \eta(T_3)e^{\beta_1 T_3}\right| e^{-\beta_1 t} \leq \frac{\epsilon}{2} \quad \text{for all} \quad t \geq T_4.$$

Then

$$\eta(t) \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1} + \epsilon \quad \text{for all} \quad t \geq T_4.$$

Hence

$$\limsup_{t\to\infty} \left(x(t) + \frac{y(t)}{\gamma_1} + \alpha' z(t)\right) \leq 1 + \frac{1}{4\beta_1} + \frac{M}{\beta_1}.$$

(iii) System (6) obviously dissipative in R^3_+. □

3 Existence and stability of equilibria

The system (6) possesses the following biological feasible equilibria.

The trivial equilibrium point $E_0(0, 0, 0)$, the axial equilibrium point $E_1(1, 0, 0)$ and the planar equilibrium point $E_2(x_2, y_2, 0)$, where $y_2 = (1 - x_2)(\alpha + x_2)$ and x_2 is given by the roots of following cubic equation:

$$\delta x^3 - \delta(1 - 2\alpha)x^2 + \big(\delta - \beta + \delta\alpha(\alpha - 2)\big)x - \alpha(\beta + \alpha\delta) = 0.$$

Remark 1. *The equilibria E_0 and E_1 exist for any parametric value, while an unique E_2 exists if $x_2 < 1$ and $\delta(1 + \alpha^2) > \beta + 2\delta\alpha$.*

We now seek the regions of parameter space for which the model system (2) admits feasible interior equilibrium (s). Any feasible equilibrium must correspond to a positive root x^* of the quadratic equation

$$g(x) = 0,$$

where

$$g(x) = x^2 + Ax + B \tag{11}$$

and A, B are given by

(i) $A = \alpha - 1$,

(ii) $B = y^* - \alpha$

for which, additionally $y^* = \frac{\gamma_2 - \alpha_1\beta_2}{\beta_2}$, and $z^* = \frac{(\alpha_1+y^*)[\gamma_1 x^* - (\beta_1+\delta y^*)(\alpha+x^*)]}{(\alpha+x^*)}$. Now, we describe the range of possibilities for which an interior positive equilibrium (equilibria) exists.

If $\gamma_2 > \alpha_1\beta_2$, then $y^* > 0$.

Case 1. If $A > 0$ and $B > 0$, then by Descartes' rule of signs there is no positive root of the equation $g(x) = 0$ and hence the model system has no interior equilibrium point.

Case 2. If $B < 0$, then by Descartes' rule of signs there exists exactly one positive root of equation (11). Also, $y^* > 0$. Now, if this positive root is denoted by x^* and satisfy the restriction $x^* > \frac{\alpha(\beta_1+\delta y^*)}{\gamma_1-\beta_1-\delta y^*}$, then $z^* > 0$. Hence if $B < 0$, and the positive root of the equation (11) is greater than $\frac{\alpha(\beta_1+\delta y^*)}{\gamma_1-\beta_1-\delta y^*}$, then there exists a unique interior equilibrium point $E^*(x^*, y^*, z^*)$.

Case 3. If $A < 0$ and $B > 0$,then by Descartes' rule of signs, equation (11) has two positive roots. For $z^* > 0$ these positive roots must be greater than $\frac{\alpha(\beta_1+\delta y^*)}{\gamma_1-\beta_1-\delta y^*}$. Also $y^* > 0$. Now, if one positive root is greater than $\frac{\alpha(\beta_1+\delta y^*)}{\gamma_1-\beta_1-\delta y^*}$ (assuming $\gamma_1 > \beta_1 + \delta y^*$), then there exists one interior equilibrium point, and if both positive roots are greater than $\frac{\alpha(\beta_1+\delta y^*)}{\gamma_1-\beta_1-\delta y^*}$, then there exists two interior equilibrium points. But if no root is greater than $\frac{\alpha(\beta_1+\delta y^*)}{\gamma_1-\beta_1-\delta y^*}$, then no interior stationary solution exists.

3.1 Local stability of the interior equilibrium point

The variational matrix for the system (6) at E^* is given by

$$J^* = \begin{bmatrix} 1 - 2x - \frac{y\alpha}{(\alpha+x)^2} & -\frac{x}{(\alpha+x)} & 0 \\ \frac{\gamma_1\alpha y}{(\alpha+x)^2} & \frac{\gamma_1 x}{(\alpha+x)} - (\beta_1 + 2\delta y) - \frac{\alpha_1 z}{(\alpha_1+y)^2} & -\frac{y}{(\alpha_1+y)} \\ 0 & \frac{\gamma_2 z^2}{(\alpha_1+y)^2} & 0 \end{bmatrix}.$$

Here,

$$\operatorname{trace} J^* = \frac{x^*y^*}{(\alpha+x^*)^2} - x^* + \frac{y^*z^*}{(\alpha_1+y^*)^2} - \delta y^*.$$

Now $\operatorname{trace} J^* < 0$ if

$$\frac{x^*y^*}{(\alpha+x^*)^2} + \frac{y^*z^*}{(\alpha_1+y^*)^2} < x^* + \delta y^*.$$

Also,

$$\det J^* = -\frac{\gamma_2(2\alpha+1)y^*z^*}{(\alpha+x^*)^2(\alpha_1+y^*)^3}\left[{x^*}^2 - px^* + qB\right],$$

where

$$p = \frac{2(y^* - \alpha^2)}{2\alpha+1}\ ,\ q = \frac{\alpha}{2\alpha+1}$$

and

$$D(J^*) = \operatorname{trace} J^* \times M(J^*) - \det J^*,$$

where $M(J^*)$ is the sum of the second order principle minors of J^*.

Therefore

$$D(J^*) = -\frac{\gamma_2 y^* {z^*}^2}{(\alpha_1 + {y^*}^2)}\left[\frac{1}{(\alpha_1+y^*)}\left(\beta_1 + 2\delta y^* + \frac{\alpha_1 z^*}{(\alpha_1+y^*)^2} - \frac{\gamma_1 x^*}{(\alpha+x)}\right)\right. \\ + \frac{\alpha\gamma_1}{(\alpha+x^*)^2}\left(\beta_1 + 2x^* + 2\delta y^* + \frac{\alpha_1 z^*}{(\alpha_1+y^*)^2}\right. \\ \left.\left. + \frac{\alpha y^*}{(\alpha+x^*)^2} - 1 - \frac{\gamma_1 x^*}{(\alpha+x^*)}\right)\right]. \tag{12}$$

To examine the local stability of the interior equilibrium (equilibria), suppose that there is one positive equilibrium $E^*(x^*, y^*, z^*)$ or two positive equilibria $E_1^*(x_1^*, y_1^*, z_1^*)$ and $E_2^*(x_2^*, y_2^*, z_2^*)$. Then (x_i^*, y_i^*, z_i^*) satisfy

$$g(x_i^*) = x_i^2 + Ax_i + B = 0$$

and

$$y_i^* = \frac{\gamma_2 - \alpha_1\beta_2}{\beta_2}, \quad z_i^* = \frac{(\alpha_1 + y_i^*)[\gamma_1 x_i^* - (\beta_1 + \delta y_i^*)(\alpha + x_i^*)]}{(\alpha + x_i^*)}, \quad i = 1, 2.$$

Let us define the function, $h(x_i^*) = {x_i^*}^2 - px_i^* + qB$, so that the signs of $h(x_i^*)$ and $\det J^*$ are opposite. It is useful to compare the two following g and h. Suppose these two function coincide at $x = \bar{x}$. Then,

$$g(\bar{x}) = h(\bar{x}),$$

or, equivalently,

$$\bar{x} = \frac{B(q-1)}{A+p}.$$

At $\bar{x}$, $g(\bar{x}) = \frac{B^2(q-1)^2}{(A+p)^2} + \frac{B(Aq+p)}{(A+p)}$.

For $A < 0$ and $B > 0$, p and q are positive and hence $g(\bar{x}) > 0$.

Case 1. If $B < 0$, recall that there exist exactly one interior equilibrium point E^*. It can easily be verified that for $B < 0$,

(i) $\operatorname{trace} J^* < 0$,

(ii) $D(J^*) = \operatorname{trace} J^* \times M(J^*) - \det J^* < 0$.

Now we shall verify the sign of $\det J^*$. For that, let $B' = -B$ so that $B' > 0$. Here two cases may arise:

a) when $p > 0$, here E^* is stable if ${x^*}^2 > px^* + qB'$, and
b) when $p < 0$, or let $p' = -p > 0$. Here E^* is stable if ${x^*}^2 + p'x^* > qB'$.

Case 2. If $A < 0$ and $B > 0$, then here also it can easily be verified that

(i) $\operatorname{trace} J^* < 0$,
(ii) $D(J^*) = \operatorname{trace} J^* \times M(J^*) - \det J^* < 0$.

For the above conditions we observe that equation (11), has two positive roots. Denote these roots by x_1^* and x_2^* with $0 < x_1^* < x_2^*$ (say), and corresponding $0 < z_1^* < z_2^*$. Assume that there exists at least one equilibrium point, so that $z_2^* > 0$. now at $x = \bar{x},\ g(\bar{x}) > 0$. Hence $0 < x_1^* < \bar{x} < x_2^*$ and therefore

$$\begin{aligned}\det J^*(x_1^*, y_1^*, z_1^*) &= -\frac{\gamma_2(2\alpha+1)y_1^* z_1^*}{(\alpha + x_1^*)^2(\alpha_1 + y_1^*)^3}h(x_1^*) \\ &> -\frac{\gamma_2(2\alpha+1)y_1^* z_1^*}{(\alpha + x_1^*)^2(\alpha_1 + y_1^*)^3}g(x_1^*) = 0.\end{aligned}$$

Hence $E_1^*(x_1^*, y_1^*, z_1^*)$ is a saddle. If we have two interior equilibria, then $0 < z_1^* < z_2^*$ and

$$\begin{aligned}\det J^*(x_2^*, y_2^*, z_2^*) &= -\frac{\gamma_2(2\alpha+1)y_2^* z_2^*}{(\alpha + x_2^*)^2(\alpha_1 + y_2^*)^3}h(x_2^*) \\ &< -\frac{\gamma_2(2\alpha+1)y_2^* z_2^*}{(\alpha + x_2^*)^2(\alpha_1 + y_2^*)^3}g(x_2^*) = 0.\end{aligned}$$

In this case, $E_2^*(x_2^*, y_2^*, z_2^*)$ is a sink.

We are now in the position to write the following theorem.

Theorem 2. *If there is exactly one interior stationary solution E^*, then it is a sink provided ${x^*}^2 - px^* + qB > 0$, where $p = \frac{2(y^* - \alpha^2)}{2\alpha+1},\ q = \frac{\alpha}{2\alpha+1}$ and $B = y^* - \alpha$. If there are two interior stationary solutions, then one is saddle and the other is sink.*

Next we shall find the conditions for the global stability of the positive equilibrium point.

3.2 Global stability

In this section we shall prove the global stability of the coexisting equilibrium point E^*. We prove the global stability result with help of a suitable Lyapunov function. Global stability of coexisting equilibrium ensures that all trajectories ultimately approaches towards the equilibrium point starting from any point within the positive octant.

Theorem 3. *Suppose the following conditions hold $y^* < \alpha(\alpha + x^*)$, $z^* < \delta\alpha_1(\alpha_1 + y^*)$, and $\gamma_2 > \beta_2(\alpha_1 + M'\gamma_1)(\alpha + y^*)$, then the coexisting equilibrium point E^* is a global attractor.*

Proof. Let us consider the following Lyapunov function,

$$V(t) = \left[x - x^* - x^* \ln \frac{x}{x^*}\right] + A_2\left[y - y^* - y^* \ln \frac{y}{y^*}\right] + A_3\left[z - z^* - z^* \ln \frac{z}{z^*}\right],$$

where A_1 and A_2 are two positive constants to be defined later. Taking the time derivative of $V(t)$ along the solution of (6), we get

$$\begin{aligned}\dot{V} = (x - x^*)\left[1 - x - \frac{y}{\alpha + x}\right] + A_2(y - y^*)\left[\frac{\gamma_1 x}{\alpha + x} - \beta_1 - \delta y - \frac{z}{\alpha_1 + y}\right] \\ + A_3(z - z^*)\left[\beta_2 z - \frac{\gamma_2 z}{\alpha_1 + y}\right].\end{aligned} \tag{13}$$

Using the following results

$$1 = x^* + \frac{y^*}{\alpha + x^*}, \quad \frac{\gamma_1 x^*}{\alpha + x^*} = \beta_1 + \delta y^* + \frac{z^*}{\alpha_1 + y^*}, \quad \beta_2 = \frac{\gamma_2}{\alpha_2 + y^*},$$

the time derivative of V becomes

$$\begin{aligned}\dot{V} = (x - x^*)\left[- (x - x^*) + \frac{y^*}{\alpha + x^*} - \frac{y}{\alpha + x}\right] \\ + A_2(y - y^*)\left[\frac{\gamma_1 x}{\alpha + x} - \frac{\gamma_1 x^*}{\alpha + x^*} - \beta_1 - \delta(y - y^*) + \frac{z^*}{\alpha_1 + y^*} - \frac{z}{\alpha_1 + y}\right] \\ + A_3(z - z^*)\left[\beta_2(z - z^*) + \frac{\gamma_2 z^*}{\alpha_1 + y^*} - \frac{\gamma_2 z}{\alpha_1 + y}\right].\end{aligned} \tag{14}$$

After some algebraic calculations, we obtain,

$$\begin{aligned}\dot{V} = (x - x^*)^2 + y^* \frac{(x - x^*)^2}{(\alpha + x^*)(\alpha + x)} - A_2\delta(y - y^*)^2 \\ + A_2 z^* \frac{(y - y^*)^2}{(\alpha_1 + y)(\alpha_1 + y^*)} + A_3\beta_2(z - z^*)^2 \\ - A_3\alpha_1\gamma_2 \frac{(z - z^*)^2}{(\alpha_1 + y)(\alpha_1 + y^*)} - A_3\gamma_2 y^* \frac{(z - z^*)^2}{(\alpha_1 + y)(\alpha_1 + y^*)} \\ + \frac{(x - x^*)(y - y^*)}{(\alpha + x)(\alpha + x^*)}\left[- \alpha - x^* + A_2\alpha\gamma_1\right] \\ + \frac{(y - y^*)(z - z^*)}{(\alpha_1 + y)(\alpha_1 + y^*)}\left[- A_2\alpha_1 - A_2 y^* + A_3\gamma_2 z^*\right].\end{aligned} \tag{15}$$

Assuming, $A_2 = \frac{\alpha + x^*}{\alpha\gamma_1}$ and $A_3 = \frac{\alpha + y^*}{\gamma_2 z^*}$, we have

$$\begin{aligned}\dot{V} = \left[- 1 + \frac{y^*}{(\alpha + x)(\alpha + x^*)}\right](x - x^*)^2 + A_2\left[- \delta + \frac{z^*}{(\alpha_1 + y)(\alpha + y^*)}\right](y - y^*)^2 \\ + A_3\left[\beta_2 - \frac{\alpha_1\gamma_2 + \gamma_2 y^*}{(\alpha_1 + y)(\alpha + y^*)}\right](z - z^*)^2.\end{aligned}$$

From Theorem 1, we observe that $\limsup_{t\to\infty}(x(t)+\frac{y(t)}{\gamma_1}) \leq M'$, where, $M' = 1+\frac{1}{4\beta_1}$. So, we can say that $x(t) < M'$ and $y(t) < M'\gamma_1$ for all time t. So, $\frac{1}{\alpha} \geq \frac{1}{\alpha+x} \geq \frac{1}{\alpha+M'}$, and $\frac{1}{\alpha_1} \geq \frac{1}{\alpha_1+y} \geq \frac{1}{\alpha_1+M'\gamma_1}$. Substituting, we get

$$\dot{V} \leq \left[-1+\frac{y^*}{\alpha(\alpha+x^*)}\right](x-x^*)^2 + A_2\left[-\delta+\frac{z^*}{\alpha_1(\alpha+y^*)}\right](y-y^*)^2 + A_3\left[\beta_2-\frac{\gamma_2}{(\alpha_1+M'\gamma_1)(\alpha+y^*)}\right](z-z^*)^2 < 0,$$

if $y^* < \alpha(\alpha+x^*)$, $z^* < \delta\alpha_1(\alpha_1+y^*)$, and $\gamma_2 > \beta_2(\alpha_1+M'\gamma_1)(\alpha+y^*)$.

Therefore by LaSalle's theorem [24], E^* is globally asymptotically stable in the xyz-space.

Hence the theorem. □

4 Numerical simulations

We began our numerical simulation with the set of parameter values taken from [1] except d. We integrate the model system (1) numerically using the MATLAB code ode45 available in the MATLAB 6.5.

We have performed the numerical simulations for different values of the parameter a_0 with $d = 0$ and keeping the other parameters fixed as given in Table 1. Results of numerical simulations show that with increase in the value of a_0, the trajectory approaches towards a strange attractor through period doubling route, starting from a periodic limit cycle. For $a_0 \geq 2.95$ the system exhibit chaotic oscillation and trajectory approaches the chaotic attractor. Projection of the phase-space trajectory on XY-plane is presented in Fig. 1 for four different values of a_0 as mentioned in the labels.

Table 1. Hypothetical parameter values taken from [1]

Parameters	b_0	v_0	d_0	v_1	a_1	d_1	d_2	v_2	v_3	c_3
Default values	0.06	1	10	2	1	10	20	0.405	1	0.038

Now we shall prove that the strange attractor shown in Fig. 1 is actually chaotic in nature. For this we will first calculate all the Lyapunov exponents associated with the strange attractor shown in Fig. 1(d). The spectrum of Lyapunov exponent is shown in Fig. 2. One can see that the largest Lyapunov exponent thus calculated is positive, showing that the strange attractor is chaotic in nature. In order to show the sensitive dependence of the trajectories on the initial conditions and to establish the divergence of two trajectories starting from two nearby points in the phase-space one can take help of the function $S(t)$ defined below. Let $X(t) \equiv (X_1(t), X_2(t), X_3(t))$ and $Y(t) \equiv (Y_1(t), Y_2(t), Y_3(t))$ are two solution trajectories of the model system (1) starting from two nearby point say x_0 and y_0 such that $||x_0 - y_0||$ is very small then S(t) stands for

$$S(t) = \sqrt{\left[(X_1(t)-Y_1(t))^2 + (X_2(t)-Y_2(t))^2 + (X_3(t)-Y_3(t))^2\right]}.$$

For detailed discussion on this function $S(t)$, please see [25]. The Plot of $S(t)$ against the time t is given in Fig. 3. It clearly shows that the difference between the two trajectories starting with two different initial point varies randomly with time. Thus we may conclude that the trajectory shown in Fig. 1(d) is chaotic in nature. The Fig. 1 is obtained by keeping $d = 0$. Next, we shall observe the role of the parameter d, on the model system (1), which we have introduced in this article.

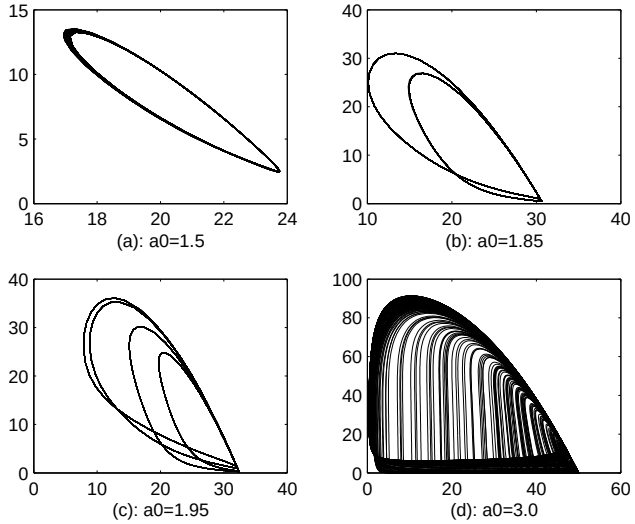

Fig. 1. Phase portrait in the XY-plane, for the system (1), showing the transition to chaos, via period-doubling from a limit cycle to strange attractors, with $d = 0$, the set of other parameter values are given in Table (1). a_0 is varied as given under each figure.

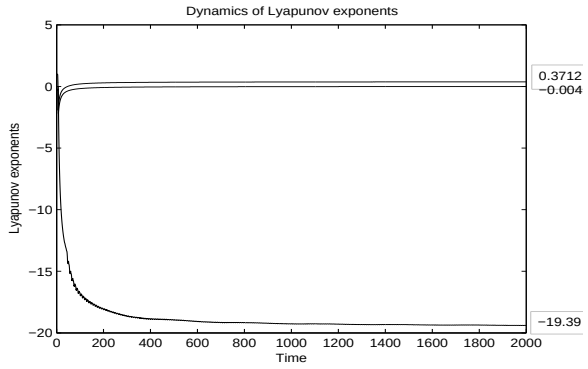

Fig. 2. The spectrum of Lyapunov exponent calculated for the strange attractor shown in Fig. 1(d).

To see the effect of intra-specific density dependence of specialist predator on the dynamics of the model system (1), we again integrate model system (1) numerically for different values of d keeping a_0 fixed at 3 and all other parameter values as mentioned in Table 1. We observe that that the chaotic nature of solution trajectory disappears with the increasing magnitude of intra-specific density dependent death rate of specialist predator (see Fig. 4). The chaos in the system can also be controled by decreasing the value of growth rate of the generalist predator c_3 (see Fig. 5).

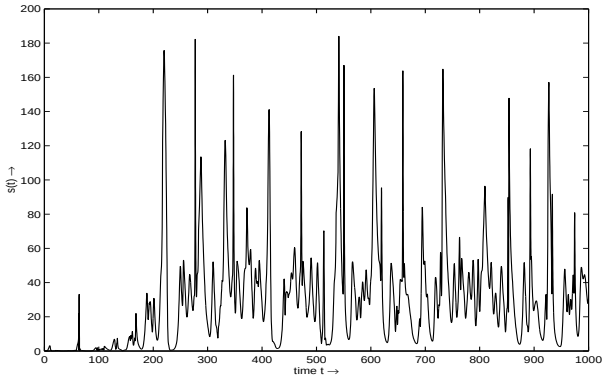

Fig. 3. separation between two nearby trajectories with advancement of time.

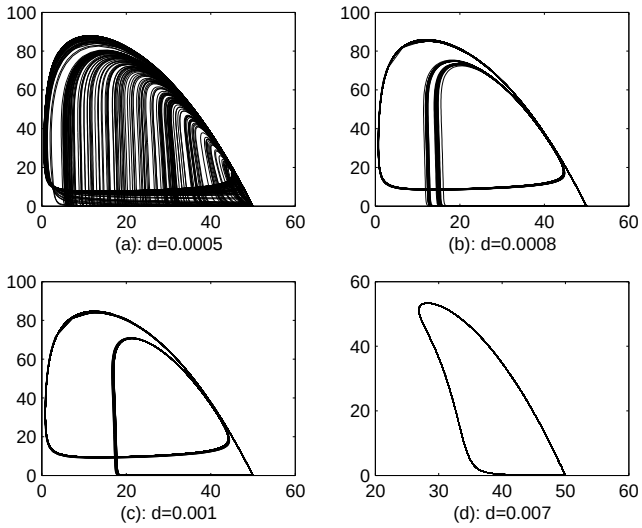

Fig. 4. Phase portrait in the XY-plane, for the system (1), showing the transition from chaos to stable period-1 limit cycle, via period-halving, with $c_3 = 0.038$ and $a_0 = 3.0$, the other parameter values are given in Table 1. d is varied as given under each figure.

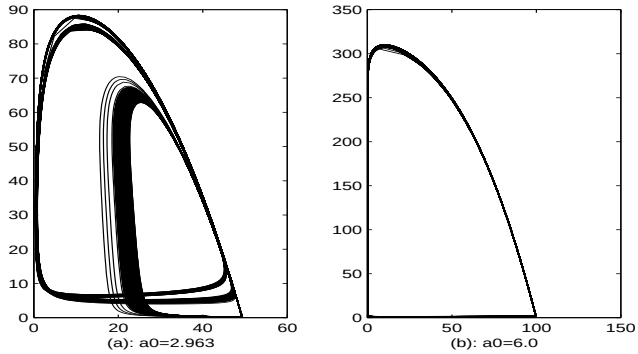

Fig. 5. Phase portrait in the XY-plane, for the system (1), showing the transition from chaos to stable period-1 limit cycle, via period-halving, with $c_3 = 0.03$ and $d = 0$, the other parameter values are given in Table 1. a_0 is varied as given under each figure.

5 Discussion

In the present work we have observed the significant change of dynamical behavior of a three species food-chain by introducing density dependent death rate of specialist predator population without altering the growth equation of the top predator. We have obtained the boundedness condition for the solutions, the local and the global stability conditions for the extended model system. In the present model we find two coexisting equilibrium point for a certain restriction on the parameters and it is interesting to note that one of which is unstable whenever other one is stable. Then we established that local asymptotic stability imply the global stability of it under some parametric restriction.

Finally, we carried out numerical simulations to substantiate the analytical findings. Our findings established the fact that the density dependent death rate for one of the predator species has ability to control the chaotic dynamics. This kind of dynamical change occur due to the reason that introduction of density dependent death rate in the growth equation of specialist predator have a negative feedback on the evolution of the model system with advancement of time. All solution trajectories starting from various points within the positive cone ultimately settle down to the coexisting equilibrium point leaving the chaotic oscillatory mode with the increasing magnitude of intra specific competition parameter indicating E^* is a global attractor. Hence, we may finally remark that the density dependent death rate of predator population has some stabilization effect on the three dimensional model system, otherwise system shows chaotic dynamics in a certain range of biological feasible parametric space.

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